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Factorization of Combinatorial *R* Matrices and Associated Cellular Automata[1]

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Solvable vertex models in statistical mechanics give rise to soliton cellular automata at $q=0$ in a ferromagnetic regime. By means of the crystal base theory we study a class of such automata associated with non-exceptional quantum affine algebras $U'_q(\hat{\mathfrak{g}}_n)$. Let B_l be the crystal of the $U'_q(\hat{\mathfrak{g}}_n)$-module corresponding to the l-fold symmetric fusion of the vector representation. For any crystal of the form $B = B_{l_1} \otimes \cdots \otimes B_{l_N}$, we prove that the combinatorial R matrix $B_M \otimes B \simeq B \otimes B_M$ is factorized into a product of Weyl group operators in a certain domain if M is sufficiently large. It implies the factorization of certain transfer matrix at $q=0$, hence the time evolution in the associated cellular automata. The result generalizes the ball-moving algorithm in the box-ball systems.

KEY WORDS: Quantum integrable systems; quantum groups; crystal bases; soliton cellular automata.

1. INTRODUCTION

The box-ball systems[24–27] are important examples of soliton cellular automata. They are discrete dynamical systems whose time evolution is expressed as a certain motion of balls along the one dimensional array of boxes. Their integrability has been understood by the ultradiscretization[28] of classical integrable systems (soliton equations). In the recent works[6, 3, 4] it was revealed that the box-ball systems may also be viewed as quantum integrable systems at $q=0$. Here by quantum integrable systems we mean the ones whose integrability is guaranteed by the Yang–Baxter equation,[1] and q is the deformation parameter in the relevant quantum group. In fact,

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the box-ball systems are identified with a $q \to 0$ limit of some two-dimensional solvable vertex models, where the role of time evolution is played by the action of a row transfer matrix. The simplest example is the original Takahashi–Satsuma automaton,[26] whose classical origin is the discrete Lotka–Volterra equation[28] and the quantum origin is the fusion six-vertex model. Here is an example of the automaton time evolution.

$$\cdots 111221121111111 \cdots$$
$$\cdots 111112212111111 \cdots$$
$$\cdots 111111212211111 \cdots$$
$$\cdots 111111121122111 \cdots$$

One regards 1 as an empty box and 2 as a box containing a ball. At each time step one moves every ball once starting from the leftmost one. The rule is that the ball goes to the nearest right empty box. One easily finds that the sequence of l balls propagate stably to the right with velocity l unless it interacts with other balls. By regarding such patterns as (ultradiscrete) solitons, the above figure illustrates how the larger soliton overtakes the smaller one with a phase shift.

In terms of the fusion six vertex model at $q = 0$, the above figure corresponds to the configuration:

$$\begin{array}{ccccccccccccccccccccc}
 & 2 & & 2 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
111 & + & 112 & + & 122 & + & 112 & + & 111 & + & 112 & + & 111 & + & 111 & + & 111 & + & 111 & + & 111 \\
 & 1 & & 1 & & 2 & & 2 & & 1 & & 2 & & 1 & & 1 & & 1 & & 1 & \\
111 & + & 111 & + & 111 & + & 112 & + & 122 & + & 112 & + & 122 & + & 112 & + & 111 & + & 111 & + & 111 \\
 & 1 & & 1 & & 1 & & 1 & & 2 & & 1 & & 2 & & 2 & & 1 & & 1 & \\
111 & + & 111 & + & 111 & + & 111 & + & 111 & + & 112 & + & 111 & + & 112 & + & 122 & + & 112 & + & 111 \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 2 & & 2 &
\end{array}$$

The left and right boundaries are to be understood as 1 or 111 everywhere. This is a configuration of the fusion six vertex model in which the quantum space is spin 1/2 (1 or 2) and the auxiliary space is spin 3/2 (111, 112, 122 or 222). At $q = 0$ only some selected vertex configurations have non-zero Boltzmann weights and the transfer matrix yields a deterministic evolution of the spins on one row to another. The vertex configurations in the above figure are the non-zero ones, and form an example of the combinatorial ($q = 0$) R matrix, which will be a main subject in this paper. Let T_M denote the row transfer matrix at $q = 0$ corresponding to the spin $M/2$ auxiliary space. The above example corresponds to T_3. Actually it can be shown that the ball-moving algorithm coincides with the action of T_M with sufficiently large M. In the above example $T_M = T_3$ holds for any $M \geqslant 2$.

The coincidence of the ultradiscrete limit of soliton equations and the $q \to 0$ limit of vertex models is an interesting phenomenon in various respects. From a statistical mechanical point of view, it roughly means that in those solvable vertex models, the profile of low-lying excitations over the ferromagnetic ground state at $q = 0$ admits an exact description by (ultradiscrete) soliton equations. From a mathematical point of view it leads to a systematic generalization[6] by means of the quantum affine algebras and the crystal base theory.[13] See Section 4 of the reference[9] for several examples of the scattering of ultradiscrete solitons.

Now we turn to a general setting, where the six vertex model is replaced with a solvable vertex model associated with the quantum affine algebra $U'_q(\hat{\mathfrak{g}}_n)$. In this paper we treat the non-exceptional case $\hat{\mathfrak{g}}_n = A^{(1)}_n$, $A^{(2)}_{2n-1}$, $A^{(2)}_{2n}$, $B^{(1)}_n$, $C^{(1)}_n$, $D^{(1)}_n$ and $D^{(2)}_{n+1}$. The box-ball systems correspond to[4] $\hat{\mathfrak{g}}_n = A^{(1)}_n$. The Boltzmann weights are trigonometric functions satisfying the Yang–Baxter equation. The row transfer matrix is specified by the auxiliary space V_M and the quantum space $\cdots \otimes V_{l_j} \otimes V_{l_{j+1}} \otimes \cdots$. Here V_M denotes the M-fold symmetric fusion of the vector representation. We suppose a ferromagnetic boundary condition, namely, the spins in the distance $|j| \gg 1$ are all equal to some prescribed element in V_{l_j}. At $q = 0$ the transfer matrix yields a deterministic evolution of the spins on one row to another.

To analyze such a situation we invoke the crystal base theory.[13] Let B_l be the crystal of V_l. It is a finite set listed in Appendix A endowed with the action of Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$: $B_l \to B_l \cup \{0\}$ for $0 \leqslant i \leqslant n$. Let $\delta_l[a] \in B_l$ be the special element as in (8). The states in the automaton are the elements $\cdots \otimes b_j \otimes b_{j+1} \otimes \cdots \in \cdots \otimes B_{l_j} \otimes B_{l_{j+1}} \otimes \cdots$ obeying the boundary condition $b_j = \delta_{l_j}[a_k]$, $|j| \gg 1$. Here a_k is specified in Table II with (11), and $k \in \mathbb{Z}$ is a label of the boundary condition at our disposal. Let T_M denote the $q = 0$ transfer matrix, or Mth time evolution in the automaton; $T_M : \cdots \otimes B_{l_j} \otimes B_{l_{j+1}} \otimes \cdots \to \cdots \otimes B_{l_j} \otimes B_{l_{j+1}} \otimes \cdots$. The map $T_M(\cdots \otimes b_j \otimes b_{j+1} \otimes \cdots) = \cdots \otimes b'_j \otimes b'_{j+1} \otimes \cdots$ is induced by the isomorphism of crystals R: $B_M \otimes B \simeq B \otimes B_M$ with $B = \cdots \otimes B_{l_j} \otimes B_{l_{j+1}} \otimes \cdots$ according to (32). We call R the *combinatorial R matrix*. It is obtained by successive applications of the elementary ones $B_M \otimes B_{l_j} \simeq B_{l_j} \otimes B_M$. The boundary condition matches the known properties like $\delta_M[a_k] \otimes \delta_{l_j}[a_k] \overset{\sim}{\mapsto} \delta_{l_j}[a_k] \otimes \delta_M[a_k]$. When M gets large, T_M stabilizes to a certain map, which we denote by T. In the box-ball system terminology ($\hat{\mathfrak{g}}_n = A^{(1)}_n$ case), this corresponds to the boxes with inhomogeneous capacities $\{l_j\}$ and the carrier with infinite capacity.

The main result of this note is Theorem 2, which states that the isomorphism $R: B_M \otimes B \simeq B \otimes B_M$ with $B = B_{l_1} \otimes \cdots \otimes B_{l_N}$ is expressed as

$$R = (\sigma_B \otimes \sigma)\, P S_{i_{k+d}} \cdots S_{i_{k+2}} S_{i_{k+1}}$$

in the domain $B_M[a_k] \otimes B \subset B_M \otimes B$ (12) if M is sufficiently large. Here S_i is the Weyl group operator[15] (4) acting on $B_M \otimes B$. $P(u \otimes v) = v \otimes u$ is the transposition, σ and σ_B are the operators corresponding to the Dynkin diagram automorphism described around Proposition 1. The above result on R reveals the factorization of the time evolution in the automaton: (Corollary 14)

$$T = \sigma_B S_{i_{k+d}} \cdots S_{i_{k+2}} S_{i_{k+1}}$$

where S_i here is the one acting on $\cdots \otimes B_{l_j} \otimes B_{l_{j+1}} \otimes \cdots$. See the explanation after Corollary 14. Such a decomposition is by no means evident from the defining relation (32). Note that T is a translation in the sense that the product $\sigma r_{i_{k+d}} \cdots r_{i_{k+1}}$ (r_i is a simple reflection) is so in the extended affine Weyl group, if σ is interpreted as the Diagram automorphism acting on the weight lattice. According to the factorization, one can consider a finer time evolution $\mathscr{T}_m$ (33) that includes the original one as $\mathscr{T}_{k+td} = T^t$ ($t \geqslant 0$). For $\hat{\mathfrak{g}}_n = A_n^{(1)}$, the change from $\mathscr{T}_m(p)$ to $\mathscr{T}_{m+1}(p)$ agrees with the original ball-moving algorithm in the box-ball systems, where one touches only the balls with a fixed color. In particular when $\forall l_j = 1$, our Definition 11 provides a representation theoretical interpretation of the earlier observation.[5] For $\hat{\mathfrak{g}}_n \neq A_n^{(1)}$, the automaton corresponding to $\forall l_j = 1$ with $a_k = 1$ has been introduced previously.[6] The formula (33) in principle provides a simple algorithm to compute the refined time evolution for general l_j and a_k in an analogous way to the ball-moving procedure for the $A_n^{(1)}$ case. The data $d \in \mathbb{Z}_{\geqslant 1}$ and $i_k \in I$ are specified in Table II. Curiously, they have stemmed from the study[18–20] of Demazure crystals.[14] It will be interesting to investigate the present result in the light of the works.[2, 16, 17, 20, 22, 23]

2. CRYSTALS

Let $\hat{\mathfrak{g}}_n = A_n^{(1)}$ ($n \geqslant 1$), $A_{2n-1}^{(2)}$ ($n \geqslant 3$), $A_{2n}^{(2)}$ ($n \geqslant 2$), $B_n^{(1)}$ ($n \geqslant 3$), $C_n^{(1)}$ ($n \geqslant 2$), $D_n^{(1)}$ ($n \geqslant 4$) and $D_{n+1}^{(2)}$ ($n \geqslant 2$). For each $\hat{\mathfrak{g}}_n$ and $l \in \mathbb{Z}_{\geqslant 1}$, the $U'_q(\hat{\mathfrak{g}}_n)$ crystal B_l has been constructed[10, 12] except for $C_n^{(1)}$ with l odd. As for $C_n^{(1)}$, B_l here is $B^{1,l}$ in the paper.[7] The finite set B_l and the actions of $\tilde{e}_i, \tilde{f}_i$: $B_l \to B_l \cup \{0\}$ for $i \in I = \{0, 1,..., n\}$ (crystal structure) have been defined. We employ the same notation as[7, 10] and quote the set B_l in Appendix A. In particular for $\hat{\mathfrak{g}}_n \neq A_n^{(1)}$, we will always assume the convention

$$x_a = \bar{x}_i, \quad \bar{x}_a = x_i, \quad \bar{a} = i \qquad \text{if} \quad a = \bar{i}, \quad 1 \leqslant i \leqslant n$$

Let us recall some other notations and the tensor product rule.

$$\varphi_i(b)=\max\{j \mid \tilde{f}_i^j b \neq 0\}, \qquad \varepsilon_i(b)=\max\{j \mid \tilde{e}_i^j b \neq 0\}$$

For two crystals B and B', the tensor product $B \otimes B'$ is defined as the set $B \otimes B' = \{b_1 \otimes b_2 \mid b_1 \in B, b_2 \in B'\}$ with the actions of $\tilde{e}_i$ and $\tilde{f}_i$ specified by

$$\tilde{e}_i(b_1 \otimes b_2)=\begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if} \quad \varphi_i(b_1) \geqslant \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if} \quad \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases} \tag{1}$$

$$\tilde{f}_i(b_1 \otimes b_2)=\begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if} \quad \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if} \quad \varphi_i(b_1) \leqslant \varepsilon_i(b_2) \end{cases} \tag{2}$$

Here $0 \otimes b$ and $b \otimes 0$ are understood to be 0. Consequently one has

$$\begin{aligned} \varphi_i(b_1 \otimes b_2) &= \varphi_i(b_2)+(\varphi_i(b_1)-\varepsilon_i(b_2))_+ \\ \varepsilon_i(b_1 \otimes b_2) &= \varepsilon_i(b_1)+(\varepsilon_i(b_2)-\varphi_i(b_1))_+ \end{aligned} \tag{3}$$

where the symbol $(x)_+$ is defined by $(x)_+=\max(x, 0)$. The Weyl group operator $S_i (i \in I)$ is defined by[15]

$$S_i b=\begin{cases} \tilde{f}_i^{\varphi_i(b)-\varepsilon_i(b)} b & \text{if} \quad \varphi_i(b) \geqslant \varepsilon_i(b) \\ \tilde{e}_i^{\varepsilon_i(b)-\varphi_i(b)} b & \text{if} \quad \varphi_i(b) \leqslant \varepsilon_i(b) \end{cases} \tag{4}$$

S_{i} satisfies the Coxeter relations.[15] For two crystals B and B', we let P: $B \otimes B' \to B' \otimes B$ denote the transposition $P(u \otimes v)=v \otimes u$. It is easy to check

$$S_i P = P S_i \qquad \text{for any} \quad i \in I \tag{5}$$

For two crystals of the form $B=B_{l_1} \otimes \cdots \otimes B_{l_N}$ and $B'=B_{l'_1} \otimes \cdots \otimes B_{l'_{N'}}$, the tensor products $B \otimes B'$ and $B' \otimes B$ are isomorphic, i.e., they have the same crystal structure. The isomorphism R: $B \otimes B' \simeq B' \otimes B$ is called the combinatorial R matrix.[11] (In this paper we do not consider the energy associated with R.) It is obtained by a successive application of the elementary ones $B_{l_i} \otimes B_{l'_j} \simeq B_{l'_j} \otimes B_{l_i}$. We will use the same symbols $\tilde{e}_i$, $\tilde{f}_i$, ε_i, φ_i, S_i, P and R irrespective of the crystals that they act.

Let Λ_i denote a fundamental weight and let $P_{cl}=\bigoplus_{i \in I} \mathbb{Z}\Lambda_i$ be the classical weight lattice. (See Section 3.1 of the paper[11] for a precise treatment.) We define a linear map σ: $P_{cl} \to P_{cl}$ as in the rightmost column of Table I. It is a Dynkin diagram automorphism. When $\hat{\mathfrak{g}}_n \neq C_n^{(1)}$ (resp. $\hat{\mathfrak{g}}_n = C_n^{(1)}$), it agrees with the one introduced after Corollary 4.6.3 of the paper[11] with $B=B_l$ (resp. $B=B_{2l}$) for any l.

Table I. The Data σ' and σ

$\hat{\mathfrak{g}}_n$	σ' on B_l	σ on P_{cl}
$A_n^{(1)}$	$a \to a-1$	$\Lambda_a \to \Lambda_{a-1}$
$A_{2n-1}^{(2)}$	$1 \leftrightarrow \bar{1}$	$\Lambda_0 \leftrightarrow \Lambda_1$
$A_{2n}^{(2)}$	id	id
$B_n^{(1)}$	$1 \leftrightarrow \bar{1}$	$\Lambda_0 \leftrightarrow \Lambda_1$
$C_n^{(1)}$	id	id
$D_n^{(1)}$	$1 \leftrightarrow \bar{1},\ n \leftrightarrow \bar{n}$	$\Lambda_0 \leftrightarrow \Lambda_1,\ \Lambda_n \leftrightarrow \Lambda_{n-1}$
$D_{n+1}^{(2)}$	id	id

We also let σ act on the index set I by the rule $i' = \sigma(i) \Leftrightarrow \sigma(\Lambda_i) = \Lambda_{i'}$. For $A_n^{(1)}$, the letters a, $a-1$ should be interpreted mod $n+1$.

We also introduce the data $d \in \mathbb{Z}_{>0}$, and the sequences $i_d,..., i_1 \in I$ and $a_d,..., a_0$ for each algebra as in Table II.

Here a_k's are taken from the letters appearing in the description of the crystals B_l in Appendix A. In the third and the fourth columns of Table II, the symbols $\left\{\begin{smallmatrix} 0,\,1 \\ 1,\,0 \end{smallmatrix}\right\}$ and $\left\{\begin{smallmatrix} 1 \\ \bar{1} \end{smallmatrix}\right\}$ mean that either the simultaneous upper choice or the simultaneous lower choice are allowed.

Proposition 1. For any $l \in \mathbb{Z}_{>0}$ the operator $\sigma' := S_{i_1} \cdots S_{i_d}: B_l \to B_l$ is a bijection having the properties:

$$\sigma' \tilde{f}_i = \tilde{f}_{\sigma(i)} \sigma', \qquad \sigma' \tilde{e}_i = \tilde{e}_{\sigma(i)} \sigma'$$

Table II. The Data d, i_k, and a_k

$\hat{\mathfrak{g}}_n$	d	$i_d,..., i_1$	$a_d,..., a_0$
$A_n^{(1)}$	n	$1, 2,..., n$	$1, 2,..., n+1$
$A_{2n-1}^{(2)}$	$2n-1$	$n-1,..., 3, 2, \left\{\begin{smallmatrix} 0,\,1 \\ 1,\,0 \end{smallmatrix}\right\}, 2, 3,..., n$	$\bar{n},..., \bar{3}, \bar{2}, \left\{\begin{smallmatrix} 1 \\ \bar{1} \end{smallmatrix}\right\}, 2,..., n-1, n, \bar{n}$
$A_{2n}^{(2)}$	$2n$	$n-1,..., 2, 1, 0, 1, 2,..., n$	$\bar{n},..., \bar{2}, \bar{1}, 1, 2,..., n, \bar{n}$
$B_n^{(1)}$	$2n-1$	$n-1,..., 3, 2, \left\{\begin{smallmatrix} 0,\,1 \\ 1,\,0 \end{smallmatrix}\right\}, 2, 3,..., n$	$\bar{n},..., \bar{3}, \bar{2}, \left\{\begin{smallmatrix} 1 \\ \bar{1} \end{smallmatrix}\right\}, 2,..., n-1, n, \bar{n}$
$C_n^{(1)}$	$2n$	$n-1,..., 2, 1, 0, 1, 2,..., n$	$\bar{n},..., \bar{2}, \bar{1}, 1, 2,..., n, \bar{n}$
$D_n^{(1)}$	$2n-2$	$n, n-2, 2,..., 2, \left\{\begin{smallmatrix} 0,\,1 \\ 1,\,0 \end{smallmatrix}\right\}, 2,..., n-2, n$	$n, \overline{n-1},..., \bar{2}, \left\{\begin{smallmatrix} 1 \\ \bar{1} \end{smallmatrix}\right\}, 2,..., n-1, \bar{n}$
$D_{n+1}^{(2)}$	$2n$	$n-1,..., 2, 1, 0, 1, 2,..., n$	$\bar{n},..., \bar{2}, \bar{1}, 1, 2,...., n, \bar{n}$

Moreover, the action of σ' is explicitly given by the second column of Table I.

The second column of Table I specifies the transformation of those letters labeling the elements of B_l. See Appendix A. For example in $A_n^{(1)}$ case, $\sigma'((x_1,..., x_{n+1})) = (x_2,..., x_{n+1}, x_1)$ in terms of the coordinates. Similarly for $A_{2n-1}^{(2)}$, $\sigma'((x_1,..., x_n, \bar{x}_n,..., \bar{x}_1)) = (\bar{x}_1, x_2,..., x_n, \bar{x}_n,..., \bar{x}_2, x_1)$. The proposition implies $\sigma' S_i = S_{\sigma(i)} \sigma'$, from which the alternative expression $S_{i_{k+1}} \cdots S_{i_d} S_{\sigma^{-1}(i_1)} \cdots S_{\sigma^{-1}(i_k)}$ is available for any $0 \leqslant k \leqslant d$ due to $S_i^2 = id$. We identify σ with σ', thereby extend the definition of its domain. Namely, we also let σ act on B_l for any l via $\sigma = S_{i_{k+1}} \cdots S_{i_d} S_{\sigma^{-1}(i_1)} \cdots S_{\sigma^{-1}(i_k)}$. (We do not exhibit l.) The result is independent of $0 \leqslant k \leqslant d$ and enjoys the properties $\sigma \tilde{f}_i = \tilde{f}_{\sigma(i)} \sigma$, and $\sigma \tilde{e}_i = \tilde{e}_{\sigma(i)} \sigma$. Proposition 1 was known[8] for some k. For the tensor product crystal $B = B_{l_1} \otimes \cdots \otimes B_{l_N}$ we write $\sigma_B = \sigma \otimes \cdots \otimes \sigma$: $B \to B$, where σ on the right side acts on each component B_{l_j} of the tensor product according to the above rule $\sigma = S_{i_{k+1}} \cdots S_{i_d} S_{\sigma^{-1}(i_1)} \cdots S_{\sigma^{-1}(i_k)}$. Obviously one has $\sigma_B \tilde{f}_i = \tilde{f}_{\sigma(i)} \sigma_B$ and $\sigma_B \tilde{e}_i = \tilde{e}_{\sigma(i)} \sigma_B$ on B, therefore

$$\sigma_B S_i = S_{\sigma(i)} \sigma_B \qquad \text{on} \quad B = B_{l_1} \otimes \cdots \otimes B_{l_N} \tag{6}$$

The combinatorial R matrix R: $B \otimes B' \simeq B' \otimes B$ satisfies

$$R(\sigma_B \otimes \sigma_{B'}) = (\sigma_{B'} \otimes \sigma_B)\, R \tag{7}$$

When acting on crystals, σ without an index shall always act on a single crystal B_l with some l.

For each $a \in \{1,..., n+1\}(\hat{\mathfrak{g}}_n = A_n^{(1)})$, $\{1,..., n, \bar{n},..., \bar{1}\}(\hat{\mathfrak{g}}_n \neq A_n^{(1)})$, we set

$$\delta_l[a] = (x_a = l,\ x_{a'} = 0 \text{ for } a' \neq a) \in B_l \tag{8}$$

with the notation in Appendix A. Using the crystal structure explicitly one easily finds

$$\delta_l[a_k] = S_{i_k}(\delta_l[a_{k-1}]) = \tilde{e}_{i_k}^{\max}(\delta_l[a_{k-1}]) \qquad 1 \leqslant k \leqslant d \tag{9}$$

$$\tilde{e}_{i_d}^{\max} \cdots \tilde{e}_{i_1}^{\max} u = \delta_l[a_d] \qquad \text{for any} \quad u \in B_l \tag{10}$$

$$\varphi_{i_k}(\delta_l[a_{k-1}]) = 0$$

for any $\hat{\mathfrak{g}}_n$, where $\tilde{e}_i^{\max} b = \tilde{e}_i^{\varepsilon_i(b)} b$. In particular (9) implies $\delta_l[a_d] = \sigma^{-1}(\delta_l[a_0])$ due to Proposition 1. Thus it is natural to extend the definition of $i_k \in I$ and a_k to all $k \in \mathbb{Z}$ by

$$i_{k+d} = \sigma^{-1}(i_k), \qquad \delta_l[a_{k+d}] = \sigma^{-1}(\delta_l[a_k]) \tag{11}$$

This way of extending the index k of a_k is independent of l, and (9) also persists for all $k \in \mathbb{Z}$. We have $\{a_k \mid k \in \mathbb{Z}\} = \{1,..., n+1\}$ for $\hat{\mathfrak{g}}_n = A_n^{(1)}$ and $\{1,..., n, \bar{n},..., \bar{1}\}$ for $\hat{\mathfrak{g}}_n \neq A_n^{(1)}$.

In this paper we will concern some asymptotic domain of the crystal B_M when M gets large. For $a \in \{a_k \mid k \in \mathbb{Z}\}$ and $M \gg 1$, we introduce the "domain" $B_M[a] \subset B_M$ by

$\hat{\mathfrak{g}}_n = A_n^{(1)}$:

$$B_M[a] = \{(u_1,..., u_{n+1}) \in B_M \mid u_a \gg u_b \text{ for any } b \in \{1,..., n+1\} \backslash \{a\}\}$$

$\hat{\mathfrak{g}}_n \neq A_n^{(1)}$:

$$B_M[a] = \left\{ (u_1,..., \bar{u}_1) \in B_M \;\middle|\; \begin{array}{l} u_a - \bar{u}_a \gg |u_b - \bar{u}_b| \quad \text{for any} \\ \quad b \in \{1,..., n\}, \quad b \neq a, \bar{a} \end{array} \right\} \tag{12}$$

In the rest of the paper we will have assertions under the conditions like $c \otimes u \otimes c' \in B \otimes B_M[a] \otimes B'$ with B and B' of the form $B = B_{l_1} \otimes \cdots \otimes B_{l_N}$, $B' = B_{l'_1} \otimes \cdots \otimes B_{l'_{N'}}$ $(N, N' \geqslant 0)$. The mathematically invalid "definition" (12) should be understood that the associated assertions are valid on condition that $M \gg 1$ and the inequalities in (12) are satisfied. Thus for example it amounts to assuming the following:

$$\begin{aligned} &\varepsilon_{i_k} \gg \varphi_{i_k} \quad \text{on} \quad B \otimes B_M[a_{k-1}] \otimes B' \\ &\varphi_{i_k} \gg \varepsilon_{i_k} \quad \text{on} \quad B \otimes B_M[a_k] \otimes B' \end{aligned} \tag{13}$$

$$\varepsilon_{i_k}(u \otimes x) = \varepsilon_{i_k}(u) \qquad \text{for} \quad u \otimes x \in B_M[a_k] \otimes B \tag{14}$$

$$\varphi_{i_{k+1}}(x \otimes u) = \varphi_{i_{k+1}}(u) \qquad \text{for} \quad x \otimes u \in B \otimes B_M[a_k] \tag{15}$$

$$\text{If} \quad u \otimes x \simeq y \otimes v \qquad \text{then} \quad u \otimes x \in B_M[a] \otimes B \Leftrightarrow y \otimes v \in B \otimes B_M[a] \tag{16}$$

In the above (13) can be checked by using the explicit formula[7, 10] for φ_i and ε_i. Equations (14) and (15) follow directly from (13) and (3). By the weight reason (16) is obvious. Moreover we may effectively treat as

$$S_{i_k}(B \otimes B_M[a_{k-1}] \otimes B') = B \otimes B_M[a_k] \otimes B' \tag{17}$$

for any $k \in \mathbb{Z}$.

3. COMBINATORIAL *R* MATRICES

Let B be any crystal of the form $B = B_{l_1} \otimes \cdots \otimes B_{l_N}$. Our goal in this section is to prove

Theorem 2. When M is sufficiently large, the combinatorial R matrix giving the isomorphism R: $B_M \otimes B \simeq B \otimes B_M$ is expressed as

$$R = (\sigma_B \otimes \sigma)\, PS_{i_{k+d}} \cdots S_{i_{k+2}} S_{i_{k+1}}$$

on $B_M[a_k] \otimes B \subset B_M \otimes B$ for any $k \in \mathbb{Z}$.

Definition 3. Let B be any crystal of the form $B = B_{l_1} \otimes \cdots \otimes B_{l_N}$. Given $u \otimes x \in B_M[a_0] \otimes B$ and $y \otimes v \in B \otimes B_M[a_0]$ we set

$$u^{|k\rangle} \otimes x^{|k\rangle} = S_{i_k} \cdots S_{i_1}(u \otimes x) \in B_M[a_k] \otimes B$$

$$y^{\langle k|} \otimes v^{\langle k|} = S_{i_k} \cdots S_{i_1}(y \otimes v) \in B \otimes B_M[a_k]$$

for $0 \leqslant k \leqslant d$.

It should be noted that $u^{|k\rangle}$ for example is not defined solely from u but only with the other element x.

It suffices to show Theorem 2 for $k = 0$. To see this, let $u \otimes x \in B[a_0] \otimes B$, $1 \leqslant k \leqslant d$ and assume $u \otimes x \overset{\sim}{\mapsto} (\sigma_B \otimes \sigma)\, PS_{i_d} \cdots S_{i_1}(u \otimes x)$ under the isomorphism $B_M \otimes B \simeq B \otimes B_M$ with M sufficiently large. Multiplying $S_{i_k} \cdots S_{i_1}$ on the both sides and using (6), one gets

$$u^{|k\rangle} \otimes x^{|k\rangle} \overset{\sim}{\mapsto} (\sigma_B \otimes \sigma)\, PS_{\sigma^{-1}(i_k)} \cdots S_{\sigma^{-1}(i_1)} S_{i_d} \cdots S_{i_{k+1}}(u^{|k\rangle} \otimes x^{|k\rangle}) \tag{18}$$

In view of (11) and (17) this proves $1 \leqslant k \leqslant d$ case. To see it for the other k, multiply $(\sigma^m \otimes \sigma_B^m)$ on the left side and $(\sigma_B^m \otimes \sigma^m)$ on the right side of (18) for an integer m and use (6) and (7).

Henceforth we shall concentrate on the $k = 0$ case of Theorem 2 in the rest of this section. Suppose $B_M[a_0] \otimes B \ni u \otimes x \overset{\sim}{\mapsto} y \otimes v \in B \otimes B_M[a_0]$ under the isomorphism $B_M \otimes B \simeq B \otimes B_M$ with M sufficiently large. Then the assertion of Theorem 2 with $k = 0$ is equivalent to

$$v = \sigma u^{|d\rangle} \tag{19}$$

$$y = \sigma_B x^{|d\rangle} \tag{20}$$

We shall prove these relations separately. In our approach, (19) can be verified directly for any choice $B = B_{l_1} \otimes \cdots \otimes B_{l_N}$. On the other hand, as for (20), we first deal with $N = 1$ case and derive N general case based on it.

Let us first treat (19).

Definition 4. For any crystal B we introduce

$$t : B \to \mathbb{Z}^d_{\geqslant 0}$$

$$b \mapsto (t_1(b),..., t_d(b))$$

$$t_k(b) = \varphi_{i_k}(\tilde{e}^{\max}_{i_{k-1}} \cdots \tilde{e}^{\max}_{i_1} b)$$

Here i_k's are those in Table II. it is easy to calculate t explicitly for $B = B_l$. For the element $b = (x_1,..., x_{n+1})$ for $A^{(1)}_n$ ($b = (x_1,..., \bar{x}_1)$ for $\hat{\mathfrak{g}}_n \neq A^{(1)}_n$), the result is summarized in

Lemma 5. The map t: $B_l \to \mathbb{Z}^d_{\geqslant 0}$ has the form:

$$t(b) = (x_n,..., x_1) \qquad \text{for} \quad A^{(1)}_n$$

$$= \left(x_n,..., x_3, x_2, \begin{Bmatrix} x_1, \bar{x}_1 \\ \bar{x}_1, x_1 \end{Bmatrix}, \bar{x}_2,..., \bar{x}_{n-1} \right) \qquad \text{for} \quad A^{(2)}_{2n-1}$$

$$= (x_n,..., x_1, x_0, \bar{x}_1,..., \bar{x}_{n-1}) \qquad \text{for} \quad A^{(2)}_{2n}$$

$$= \left(2x_n + x_0, x_{n-1},..., x_2, \begin{Bmatrix} x_1, \bar{x}_1 \\ \bar{x}_1, x_1 \end{Bmatrix}, \bar{x}_2,..., \bar{x}_{n-1} \right) \qquad \text{for} \quad B^{(1)}_n$$

$$= (x_n,..., x_1, x_0, \bar{x}_1,..., \bar{x}_{n-1}) \qquad \text{for} \quad C^{(1)}_n$$

$$= \left(x_n + x_{n-1}, x_{n-2},..., x_2, \begin{Bmatrix} x_1, \bar{x}_1 \\ \bar{x}_1, x_1 \end{Bmatrix}, \bar{x}_2,..., \bar{x}_{n-2}, \bar{x}_{n-1} + x_n \right)$$

$$\text{for} \quad D^{(1)}_n$$

$$= (2x_n + x_0, x_{n-1},..., x_1, x_\varnothing, \bar{x}_1,..., \bar{x}_{n-1}) \qquad \text{for} \quad D^{(2)}_{n+1}$$

See Appendix A for the notation x_0 in $A^{(2)}_{2n}$, $C^{(1)}_n$ and $x_\varnothing$ in $D^{(2)}_{n+1}$. The upper and lower choices correspond to those in Table II. From Lemma 5 we derive a useful fact.

Proposition 6. the map t: $B_l \to \mathbb{Z}^d_{\geqslant 0}$ is injective for any $l \in \mathbb{Z}_{\geqslant 1}$.

Lemma 7. Under Definition 3 one has

$$\varepsilon_{i_k}(u^{|k\rangle}) = \varphi_{i_k}(u^{|k-1\rangle} \otimes x^{|k-1\rangle}), \qquad 1 \leqslant k \leqslant d \tag{21}$$

Proof. Definition 3 tells $u^{|k\rangle} \otimes x^{|k\rangle} = S_{i_k}(u^{|k-1\rangle} \otimes x^{|k-1\rangle})$. Since $u^{|k-1\rangle} \otimes x^{|k-1\rangle} \in B_M[a_{k-1}] \otimes B$, this S_{i_k} acts as

$$
\begin{aligned}
u^{|k\rangle} \otimes x^{|k\rangle} &= \tilde{e}_{i_k}^{q} u^{|k-1\rangle} \otimes \tilde{e}_{i_k}^{q'} x^{|k-1\rangle} \\
q &= \varepsilon_{i_k}(u^{|k-1\rangle}) - \varphi_{i_k}(x^{|k-1\rangle}) - (\varphi_{i_k}(u^{|k-1\rangle}) - \varepsilon_{i_k}(x^{|k-1\rangle}))_+ \qquad (22) \\
q' &= (\varepsilon_{i_k}(x^{|k-1\rangle}) - \varphi_{i_k}(u^{|k-1\rangle}))_+
\end{aligned}
$$

Thus we have $\varepsilon_{i_k}(u^{|k\rangle}) = \varepsilon_{i_k}(u^{|k-1\rangle}) - q \overset{(3)}{=} \varphi_{i_k}(u^{|k-1\rangle} \otimes x^{|k-1\rangle})$. ∎

For integers p, q depending on M in general, we write $p \lesssim q$ to mean $p < q$ or $p - q = M$-independent for $M \gg 1$.

Lemma 8. Let $u \in B_M[a_0] \subset B_M$ for sufficiently large M. For $(c_1, \ldots, c_d) \in \mathbb{Z}_{\geqslant 0}^d$, define $u^{\langle k\rangle}$ $(0 \leqslant k \leqslant d)$ by $u^{\langle k\rangle} = \tilde{e}_{i_k}^{\varepsilon_{i_k}(u^{\langle k-1\rangle}) - c_k} u^{\langle k-1\rangle}$ $(1 \leqslant k \leqslant d)$ and $u^{\langle 0\rangle} = u$. Suppose $c_j \lesssim t_j(u)$ for all $1 \leqslant j \leqslant d$. Then the following hold for $1 \leqslant k \leqslant d$:

$$\varphi_{i_k}(u^{\langle k-1\rangle}) = t_k(u) \tag{23}$$

$$\varepsilon_{i_k}(u^{\langle k\rangle}) = c_k \tag{24}$$

$$t_k(\sigma u^{\langle d\rangle}) = c_k \tag{25}$$

Although $u^{\langle k\rangle}$ depends on c_j's, the right side of (23) is independent of them. Actually (24) is trivial. The other relations in the lemma can be verified with a direct calculation by using the crystal structure of B_M. Under Definition 3, Lemma 8 immediately leads to $(1 \leqslant k \leqslant d)$

$$\varphi_{i_k}(u^{|k-1\rangle}) = t_k(u) \tag{26}$$

$$\varphi_{i_k}(v^{\langle k-1|}) = t_k(v) \tag{27}$$

$$\varepsilon_{i_k}(u^{|k\rangle}) = t_k(\sigma u^{|d\rangle}) \tag{28}$$

The right sides of (26) and (27) are independent of x and y in Definition 3, respectively.

Proof of (19). Suppose $B_M[a_0] \otimes B \ni u \otimes x \overset{\sim}{\mapsto} y \otimes v \in B \otimes B_M[a_0]$ under the isomorphism $B_M \otimes B \simeq B \otimes B_M$ with M sufficiently large. We employ the notation in Definition 3. Applying $S_{i_{k-1}} \cdots S_{i_1}$ to the both sides of $u \otimes x \overset{\sim}{\mapsto} y \otimes v$, one gets $u^{|k-1\rangle} \otimes x^{|k-1\rangle} \overset{\sim}{\mapsto} y^{\langle k-1|} \otimes v^{\langle k-1|}$, therefore $\varphi_{i_k}(u^{|k-1\rangle} \otimes x^{|k-1\rangle}) = \varphi_{i_k}(y^{\langle k-1|} \otimes v^{\langle k-1|})$. But

$$
\begin{aligned}
\varphi_{i_k}(u^{|k-1\rangle} \otimes x^{|k-1\rangle}) &\overset{(21)}{=} \varepsilon_{i_k}(u^{|k\rangle}) \overset{(28)}{=} t_k(\sigma u^{|d\rangle}) \\
\varphi_{i_k}(y^{\langle k-1|} \otimes v^{\langle k-1|}) &\overset{(15)}{=} \varphi_{i_k}(v^{\langle k-1|}) \overset{(27)}{=} t_k(v)
\end{aligned}
$$

are valid for $1 \leqslant k \leqslant d$, telling that $t(v) = t(\sigma u^{|d\rangle})$. Thus (19) follows from Proposition 6. ∎

Now we proceed to the proof of (20) with the simple choice $B = B_l$.

Lemma 9. Suppose $B_M[a_0] \otimes B_l \ni \delta_M[a_0] \otimes z \overset{\sim}{\mapsto} \tilde{z} \otimes \tilde{u} \in B_l \otimes B_M[a_0]$ under the isomorphism $B_M \otimes B_l \simeq B_l \otimes B_M$ with M sufficiently large. Then we have $t(\tilde{u}) = t(z)$.

Proof. Define $u^{|k\rangle} \otimes z^{|k\rangle}$ by Definition 3 starting from $u \otimes z = u^{|0\rangle} \otimes z^{|0\rangle} = \delta_M[a_0] \otimes z$. From (19) we already know that $\tilde{u} = \sigma u^{|d\rangle}$. Thus we have

$$\begin{aligned} t_k(\tilde{u}) &= t_k(\sigma u^{|d\rangle}) \overset{(28)}{=} \varepsilon_{i_k}(u^{|k\rangle}) \overset{(21)}{=} \varphi_{i_k}(u^{|k-1\rangle} \otimes z^{|k-1\rangle}) \\ &\overset{(3)}{=} \varphi_{i_k}(z^{|k-1\rangle}) + (\varphi_{i_k}(u^{|k-1\rangle}) - \varepsilon_{i_k}(z^{|k-1\rangle}))_+ \\ &\overset{(26)}{=} \varphi_{i_k}(z^{|k-1\rangle}) + (t_k(u) - \varepsilon_{i_k}(z^{|k-1\rangle}))_+ \end{aligned}$$

Note from the explicit forms in Lemma 5 that $t_k(u) = t_k(\delta_M[a_0]) = 0$, from which $t_k(\tilde{u}) = \varphi_{i_k}(z^{|k-1\rangle})$ follows. In view of $\varphi_{i_k}(u^{|k-1\rangle}) \overset{(26)}{=} t_k(u) = 0$ and (13), it also follows that $z^{|k\rangle} = \tilde{e}_{i_k}^{\max} \cdots \tilde{e}_{i_1}^{\max} z$. Therefore we conclude $\varphi_{i_k}(z^{|k-1\rangle}) = t_k(z)$ from Definition 4. ▮

Lemma 10. Given $y \otimes v \in B_l \otimes B_M[a_0]$, set

$$y^{(k)} \otimes v^{(k)} = \tilde{e}_{i_k}^{\max} \cdots \tilde{e}_{i_1}^{\max}(y \otimes v)$$

for $0 \leqslant k \leqslant d$. Then we have $t(\sigma v^{(d)}) = t(y)$.

Proof. Since $y^{(k)} = \tilde{e}_{i_k}^{\max} \cdots \tilde{e}_{i_1}^{\max} y$, we have

$$v^{(k)} = \tilde{e}_{i_k}^{\varepsilon_{i_k}(v^{(k-1)}) - \varphi_{i_k}(y^{(k-1)})} v^{(k-1)} = \tilde{e}_{i_k}^{\varepsilon_{i_k}(v^{(k-1)}) - t_k(y)} v^{(k-1)}$$

Applying (25) we obtain $t(\sigma v^{(d)}) = t(y)$. ▮

Proof of (20) for $\mathsf{B} = \mathsf{B}_l$. Suppose $B_M[a_0] \otimes B_l \ni u \otimes x \overset{\sim}{\mapsto} y \otimes v \in B_l \in B_M[a_0]$ under the isomorphism $B_M \otimes B_l \simeq B_l \otimes B_M$ with M sufficiently large. We employ the notation in Definition 3. Applying $(\sigma \otimes \sigma)\, \tilde{e}_{i_d}^{\max} \cdots \tilde{e}_{i_1}^{\max}$ to the both sides and using (10), one finds

$$B_M[a_0] \otimes B_l \ni \delta_M[a_0] \otimes \sigma x^{|d\rangle} \overset{\sim}{\mapsto} (\sigma \otimes \sigma)\, \tilde{e}_{i_d}^{\max} \cdots \tilde{e}_{i_1}^{\max}(y \otimes v) \in B_l \otimes B_M[a_0]$$

Setting

$$\delta_M[a_0] \otimes \sigma x^{|d\rangle} \overset{\sim}{\mapsto} \tilde{z} \otimes \tilde{u} \in B_l \otimes B_M[a_0]$$

$$(\sigma \otimes \sigma)\, \tilde{e}_{i_d}^{\max} \cdots \tilde{e}_{i_1}^{\max}(y \otimes v) = (\sigma y^{(d)}) \otimes (\sigma v^{(d)})$$

we have $\tilde{z} = \sigma y^{(d)}$ and $\tilde{u} = \sigma v^{(d)}$. Thus $t(\tilde{u}) = t(\sigma v^{(d)})$. But we know $t(\tilde{u}) = t(\sigma x^{|d\rangle})$ from Lemma 9 and $t(\sigma v^{(d)}) = t(y)$ from Lemma 10. Therefore $t(\sigma x^{|d\rangle}) = t(y)$, compelling $y = \sigma x^{|d\rangle}$ by Proposition 6. ∎

To show (20) for the general choice $B = B_{l_1} \otimes \cdots \otimes B_{l_N}$, we prepare

Definition 11. For $s, s' \in \mathbb{Z}_{\geqslant 0}$, $b, b' \in B_l$ and $i \in I$, we let the vertex diagram

$$\begin{array}{ccc} & b & \\ & | & \\ s & \overset{i}{-\!\!\!-\!\!\!+\!\!\!-\!\!\!-} & s' \\ & | & \\ & b' & \end{array}$$

denote the relations

$$b' = \tilde{e}_i^{(\varepsilon_i(b) - s)_+} b, \qquad s' = \varphi_i(b) + (s - \varepsilon_i(b))_+$$

Here l can be any positive integer but we do not exhibit it in the diagram. The color $i \in I$ is attached to the horizontal line. The diagram should not be confused with the one representing the combinatorial R matrix.[4] Given i, (b', s') is uniquely fixed from (b, s). Thus for example the diagram

$$\begin{array}{ccccccccc} & b_1 & & b_2 & & & & b_N & \\ & | & & | & & & & | & \\ s_0 & \overset{i}{-\!\!\!-\!\!\!+\!\!\!-\!\!\!-} & s_1 & \overset{i}{-\!\!\!-\!\!\!+\!\!\!-\!\!\!-} & s_2 & -\cdots- & s_{N-1} & \overset{i}{-\!\!\!-\!\!\!+\!\!\!-\!\!\!-} & s_N \\ & | & & | & & & & | & \\ & b'_1 & & b'_2 & & & & b'_N & \end{array}$$

is uniquely determined from s_0, i and $b_1 \otimes \cdots \otimes b_N$. From Definition 11 it implies

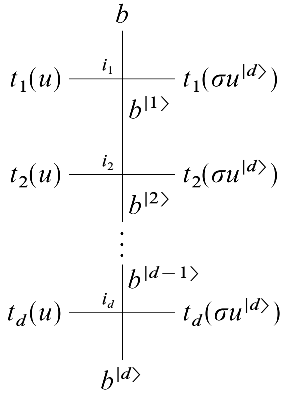
$$b'_1 \otimes \cdots \otimes b'_N = \tilde{e}_i^{(\varepsilon_i(b_1 \otimes \cdots \otimes b_N) - s_0)_+}(b_1 \otimes \cdots \otimes b_N) \tag{29}$$

Having established Theorem 2 for $B = B_l$, we already know that

$$B_M[a_0] \otimes B_l \ni u \otimes b \stackrel{\sim}{\mapsto} \sigma b^{|d\rangle} \otimes \sigma u^{|d\rangle} \in B_l \otimes B_M[a_0]$$

under the isomorphism $B_M \otimes B_l \simeq B_l \otimes B_M$ for M sufficiently large, where $u^{|k\rangle} \otimes b^{|k\rangle} = S_{i_k} \cdots S_{i_1}(u \otimes b)$ $(0 \leqslant k \leqslant d)$.

Proposition 12. Under the above stated setting, the following diagram holds.



Proof. By Definition 11 we are to check

$$b^{|k\rangle} = \tilde{e}_{i_k}^{(\varepsilon_{i_k}(b^{|k-1\rangle}) - t_k(u))_+} b^{|k-1\rangle}$$

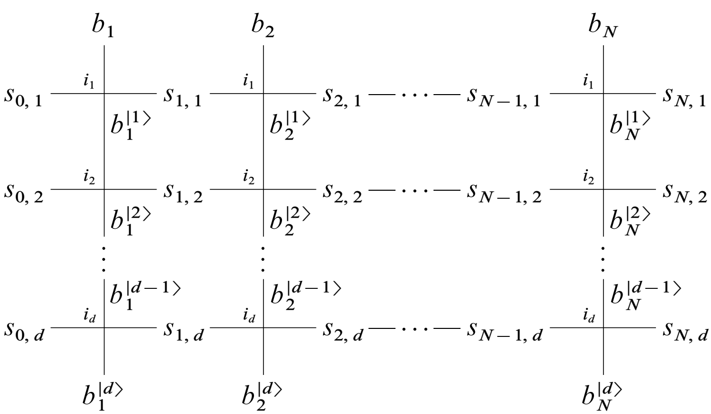
$$t_k(\sigma u^{|d\rangle}) = \varphi_{i_k}(b^{|k-1\rangle}) + (t_k(u) - \varepsilon_{i_k}(b^{|k-1\rangle}))_+$$

for $1 \leqslant k \leqslant d$. To show the former, set $x = b$ and apply (26) in q' appearing in (22). The left side of the latter reads

$$\begin{aligned} t_k(\sigma u^{|d\rangle}) &\overset{(28)}{=} \varepsilon_{i_k}(u^{|k\rangle}) \overset{(21)}{=} \varphi_{i_k}(u^{|k-1\rangle} \otimes b^{|k-1\rangle}) \\ &\overset{(3)}{=} \varphi_{i_k}(b^{|k-1\rangle}) + (\varphi_{i_k}(u^{|k-1\rangle}) - \varepsilon_{i_k}(b^{|k-1\rangle}))_+ \end{aligned}$$

which is equal to the right side owing to (26). ∎

Proof of (20) for $\mathsf{B} = \mathsf{B}_{\mathsf{l}_1} \otimes \cdots \otimes \mathsf{B}_{\mathsf{l}_N}$. Given any $x = b_1 \otimes \cdots \otimes b_N \in B$ and $u \in B_M[a_0]$, set $s_{0,k} = t_k(u)$ $(1 \leqslant k \leqslant d)$. Let $b_j^{|k\rangle} \in B_{l_j}$, $s_{j,k} \in \mathbb{Z}_{\geqslant 0}$ $(1 \leqslant k \leqslant d,\ 1 \leqslant j \leqslant N)$ be the ones uniquely determined from the diagram:



By a repeated use of Proposition 12, one has

$$u \otimes b_1 \otimes \cdots \otimes b_N \overset{\sim}{\mapsto} \sigma b_1^{|d\rangle} \otimes \sigma b_2^{|d\rangle} \otimes \cdots \otimes \sigma b_N^{|d\rangle} \otimes v$$
$$= (\sigma_B(b_1^{|d\rangle} \otimes \cdots \otimes b_N^{|d\rangle})) \otimes v$$

under the isomorphism $B_M \otimes B \simeq B \otimes B_M$ with sufficiently large M. On the other hand we introduce $u^{|k\rangle} \otimes x^{|k\rangle} := S_{i_k} \cdots S_{i_1}(u \otimes x) \in B_M[a_k] \otimes B$. (Although not necessary here, $v \in B_M[a_0]$ is uniquely determined from $t(v) = (s_{N,1},..., s_{N,d})$ by Proposition 6, and we already know that the result coincides with $v = \sigma u^{|d\rangle}$ from (19).) We are to show $x^{|d\rangle} = b_1^{|d\rangle} \otimes \cdots \otimes b_N^{|d\rangle}$. In fact, $x^{|k\rangle} = b_1^{|k\rangle} \otimes \cdots \otimes b_N^{|k\rangle}$ can be proved by induction on k as follows. (We set $b_1^{|0\rangle} \otimes \cdots \otimes b_N^{|0\rangle} = b_1 \otimes \cdots \otimes b_N$.) It is obvious for $k = 0$. From (29) we have

$$b_1^{|k\rangle} \otimes \cdots \otimes b_N^{|k\rangle} = \tilde{e}_{i_k}^{(m - s_{0,k})_+}(b_1^{|k-1\rangle} \otimes \cdots \otimes b_N^{|k-1\rangle}) \tag{30}$$

where $m = \varepsilon_{i_k}(b_1^{|k-1\rangle} \otimes \cdots \otimes b_N^{|k-1\rangle})$. On the other hand, $x^{|k\rangle}$ is determined from the recursion relation:

$$x^{|k\rangle} = \tilde{e}_{i_k}^{(\varepsilon_{i_k}(x^{|k-1\rangle}) - \varphi_{i_k}(u^{|k-1\rangle}))_+} x^{|k-1\rangle} \tag{31}$$

because of $\varepsilon_{i_k}(u^{|k-1\rangle}) \gg 1$. Note that $\varphi_{i_k}(u^{|k-1\rangle}) \overset{(26)}{=} t_k(u) = s_{0,k}$. Thus the two recursion relations (30) and (31) lead to $x^{|k\rangle} = b_1^{|k\rangle} \otimes \cdots \otimes b_N^{|k\rangle}$ under the induction assumption $x^{|k-1\rangle} = b_1^{|k-1\rangle} \otimes \cdots \otimes b_N^{|k-1\rangle}$. ∎

We have finished the proof of (20) for any $B = B_{l_1} \otimes \cdots \otimes B_{l_N}$, and thereby the proof of Theorem 2.

Example 13. Consider $\hat{\mathfrak{g}}_n = A_3^{(1)}$ $(d = 3)$. The data in Table II reads $i_j \equiv a_j \equiv 4 - j \bmod 4$ with $i_j \in \{0, 1, 2, 3\}$ and $a_j \in \{1, 2, 3, 4\}$. To save the space the element $(x_1, x_2, x_3, x_4) = (3, 2, 1, 0) \in B_6$ is denoted by 111223 for example. Then one has $\sigma(111223) = 112444$ according to Table I.

Let us take $b \otimes c = 111223 \otimes 344 \in B_6 \otimes B_3$ and seek its image under the isomorphism R: $B_6 \otimes B_3 \overset{\sim}{\to} B_3 \otimes B_6$. It is known[21] that $R(b \otimes c) = 223 \otimes 111344$. This can be derived by taking $N = 1$, $M = 6$ and $k = 3$ in Theorem 2, which reads $R = (\sigma \otimes \sigma) PS_2S_3S_0$. Namely we may regard $b \otimes c \in B_M[1] \otimes B_3$, where $M = 6$ and $x_1 = 3$ for b are already sufficiently large so that

$$
\begin{aligned}
b\otimes c = 111223\otimes 344 &\xmapsto{S_0} 112234\otimes 344\\
&\xmapsto{S_3} 112234\otimes 334\\
&\xmapsto{S_2} 112224\otimes 334\\
&\xmapsto{P} 334\otimes 112224 \xmapsto{\sigma\otimes\sigma} 223\otimes 111344 = R(b\otimes c)
\end{aligned}
$$

For comparison, take a smaller example $R(11223\otimes 344) = 223\otimes 11344 \in B_3\otimes B_5$. Theorem 2 is not applicable in this case under any choice of k because $S_i(11223\otimes 344) = 11223\otimes 344$ for any $i\in\{0, 1, 2, 3\}$ and $(\sigma\otimes\sigma)\, P(11223\otimes 344) = 233\otimes 11244 \neq 223\otimes 11344$.

4. CELLULAR AUTOMATA

The factorization of the combinatorial R matrix shown in Theorem 2 induces the factorization of the time evolution of the associated cellular automaton. Consider the isomorphism

$$B_M\otimes(\cdots\otimes B_{l_j}\otimes B_{l_{j+1}}\otimes\cdots)\simeq(\cdots\otimes B_{l_j}\otimes B_{l_{j+1}}\otimes\cdots)\otimes B_M$$

induced by the successive application of combinatorial R matrix $B_M\otimes B_{l_j}\simeq B_{l_j}\otimes B_M$. We impose the boundary condition on $b_j\in B_{l_j}$ as $b_j=\delta_{l_j}[a_k]$ for $|j|\gg 1$, where the choice of $k\in\mathbb{Z}$ is arbitrary. Assume the following properties:

(i) $\delta_M[a_k]\otimes\delta_l[a_k]\overset{\sim}{\mapsto}\delta_l[a_k]\otimes\delta_M[a_k]$ for any M, l,

(ii) $u\otimes\delta_{l_j}[a_k]\otimes\delta_{l_{j+1}}[a_k]\otimes\cdots\otimes\delta_{l_{j+N}}[a_k]\overset{\sim}{\mapsto}\tilde{b}_j\otimes\cdots\otimes\tilde{b}_{j+N}\otimes\delta_M[a_k]$ for any $u\in B_M$ if N is sufficiently large,

where $\tilde{b}_{l_j}\otimes\cdots\otimes\tilde{b}_{l_{j+N}}$ is some element in $B_{l_j}\otimes\cdots\otimes B_{l_{j+N}}$. ((i) is indeed valid by the weight reason.) Then under the isomorphism $B_M\otimes(\cdots\otimes B_{l_j}\otimes B_{l_{j+1}}\otimes\cdots)\simeq(\cdots\otimes B_{l_j}\otimes B_{l_{j+1}}\otimes\cdots)\otimes B_M$,

$$\delta_M[a_k]\otimes(\cdots\otimes b_j\otimes b_{j+1}\otimes\cdots)\overset{\sim}{\mapsto}(\cdots\otimes b'_j\otimes b'_{j+1}\otimes\cdots)\otimes\delta_M[a_k] \tag{32}$$

is valid for some b'_j's. One may regard the system as an automaton which undergoes the time evolution $p=\cdots\otimes b_j\otimes b_{j+1}\otimes\cdots\mapsto p'=\cdots\otimes b'_j\otimes b'_{j+1}\otimes\cdots$. When M gets large, the transformation $p\mapsto p'$ stabilizes to a certain map, which we denote[6] by $T(p)=p'$. By taking $B=\cdots\otimes B_{l_j}\otimes B_{l_{j+1}}\otimes\cdots$ in Theorem 2 and using (6), (11) we obtain

Corollary 14. Under the boundary condition $b_j = \delta_{l_j}[a_k] \in B_{l_j}$, for $|j| \gg 1$, the time evolution T acts on $B = \cdots \otimes B_{l_j} \otimes B_{l_{j+1}} \otimes \cdots$ as

$$T^t = \begin{cases} \sigma_B^t S_{i_{k+td}} \cdots S_{i_{k+2}} S_{i_{k+1}} & \text{if} \quad t \in \mathbb{Z}_{\geqslant 0} \\ \sigma_B^t S_{i_{k+td+1}} \cdots S_{i_{k-1}} S_{i_k} & \text{if} \quad t \in \mathbb{Z}_{<0} \end{cases}$$

Actually all the Weyl group operators S_{i_m} for $t > 0$ (resp. $t < 0$) in the above act as $S_{i_m} = \tilde{e}_{i_m}^{\max}$ (resp. $S_{i_m} = \tilde{f}_{i_m}^{\max}$) on B since they always hit such states $p \in B$ that $\varepsilon_{i_m}(p) \gg 1$ (resp. $\varphi_{i_m}(p) \gg 1$). Corollary 14 exhibits a factorization of the time evolution of the automaton having the background (vacuum) configuration $\cdots \otimes \delta_{l_j}[a_k] \otimes \delta_{l_{j+1}}[a_k] \otimes \cdots$ specified by $k \in \mathbb{Z}$. When the partial factor $S_{i_m} S_{i_{m-1}} \cdots S_{i_{k+1}}$ $(k+d \geqslant m \geqslant k+1)$ in T is applied, the background, hence the boundary condition, changes into $\cdots \otimes \delta_{l_j}[a_m] \otimes \delta_{l_{j+1}}[a_m] \otimes \cdots$ according to (9). A generalization of T that does not change the background in every intermediate step is constructed as follows:

$$\begin{aligned} \mathscr{T}_m &= \sigma_{B,k,m} S_{i_m} \cdots S_{i_{k+2}} S_{i_{k+1}}, \qquad m \geqslant k+1 \\ \sigma_{B,k,m} &= \cdots \otimes \sigma_{k,m} \otimes \sigma_{k,m} \otimes \cdots \\ \sigma_{k,m} &= S_{i_{k+1}} S_{i_{k+2}} \cdots S_{i_m} \end{aligned} \tag{33}$$

where $\sigma_{k,m}$ acts on each component B_{l_j} of the tensor product. We understand $\mathscr{T}_k = id$. For any $m \geqslant k$, the operator $\mathscr{T}_m$ retains the background in the original form $\cdots \otimes \delta_{l_j}[a_k] \otimes \delta_{l_{j+1}}[a_k] \otimes \cdots$ due to (9). Moreover from Corollary 14 we find that the original evolution under t-time step T^t is included in $\{\mathscr{T}_m\}$ as $T^t = \mathscr{T}_{k+td}$ $(t \geqslant 0)$. For $A_n^{(1)}$ the evolution of the state p according to $p = \mathscr{T}_k(p), \mathscr{T}_{k+1}(p), \mathscr{T}_{k+2}(p), \ldots$ agrees with that obtained by the ball-moving algorithm in the box-ball systems[24, 25, 27] under a convention adjustment. In particular for $A_1^{(1)}$, the evolution rule in terms of the Dyck language[29] essentially agrees with the crystal theoretic signature rule in applying $S_i = \tilde{e}_i^{\max}$ or $\tilde{f}_i^{\max}$ $(i = 0, 1)$ explained in Section 1.3 of the reference.[18]

Example 15. Consider $\hat{\mathfrak{g}}_n = A_5^{(2)}$ $(d = 5)$. To save the space the element $(x_1, x_2, x_3, \bar{x}_3, \bar{x}_2, \bar{x}_1) = (2, 1, 2, 1, 0, 1) \in B_7$ is denoted by $1123\bar{3}\bar{3}\bar{1}$ for example. Let us take $k = 0$, hence the initial background is $\cdots \otimes \delta_{l_j}[\bar{3}] \otimes \delta_{l_{j+1}}[\bar{3}] \otimes \cdots$. We employ $i_5, \ldots, i_1 = 2, 0, 1, 2, 3$ and $a_5, \ldots, a_0 = \bar{3}, \bar{2}, 1, 2, 3, \bar{3}$ correspondingly. Take

$$\begin{aligned} p &= \cdots \bar{3}\bar{3} \cdot 1\bar{2} \cdot 3 \cdot \bar{3}\bar{1} \cdot 2 \cdot \bar{3}\bar{3} \cdot \bar{3}\bar{3} \cdots \\ &\in \cdots \otimes B_2 \otimes B_2 \otimes B_1 \otimes B_2 \otimes B_1 \otimes B_2 \otimes B_2 \otimes \cdots \end{aligned}$$

where $\cdot$ stands for $\otimes$ and $\cdots$ parts on the both ends represent the configuration identical with the background. Then the time evolution (32) is given by

$$T(p) = \cdots \overline{33} \cdot \overline{33} \cdot 1 \cdot \overline{32} \cdot \bar{1} \cdot 23 \cdot \overline{33} \cdots$$

According to Corollary 14, the evolution is decomposed into the following steps.

$$\begin{aligned}
p &= \cdots \overline{33} \cdot 1\bar{2} \cdot 3 \cdot \overline{31} \cdot 2 \cdot \overline{33} \cdot \overline{33} \cdots \\
S_3(p) &= \cdots 33 \cdot 1\bar{2} \cdot 3 \cdot \overline{31} \cdot 2 \cdot 33 \cdot 33 \cdots \\
S_2 S_3(p) &= \cdots 22 \cdot 1\bar{3} \cdot 2 \cdot \overline{31} \cdot 2 \cdot 33 \cdot 22 \cdots \\
S_1 S_2 S_3(p) &= \cdots 11 \cdot 1\bar{3} \cdot 2 \cdot \overline{32} \cdot 1 \cdot 33 \cdot 11 \cdots \\
S_0 S_1 S_2 S_3(p) &= \cdots \overline{22} \cdot \overline{32} \cdot \bar{1} \cdot \overline{32} \cdot 1 \cdot 33 \cdot \overline{22} \cdots \\
S_2 S_0 S_1 S_2 S_3(p) &= \cdots \overline{33} \cdot \overline{33} \cdot \bar{1} \cdot \overline{32} \cdot 1 \cdot 23 \cdot \overline{33} \cdots \\
\sigma_B S_2 S_0 S_1 S_2 S_3(p) &= \cdots \overline{33} \cdot \overline{33} \cdot 1 \cdot \overline{32} \cdot \bar{1} \cdot 23 \cdot \overline{33} \cdots = T(p)
\end{aligned}$$

where in the last step we used the fact that σ_B interchanges the letters 1 and $\bar{1}$ in each component as specified in Table I. Here the background configuration is changing except the last step.

Alternatively the evolution may also be decomposed according to (33) into the following steps, in which the original background $\cdots \otimes \delta_{l_j}[\bar{3}] \otimes \delta_{l_{j+1}}[\bar{3}] \otimes \cdots$ is kept unchanged.

$$\begin{aligned}
p = \mathscr{T}_0(p) &= \cdots \overline{33} \cdot 1\bar{2} \cdot 3 \cdot \overline{31} \cdot 2 \cdot \overline{33} \cdot \overline{33} \cdots \\
\mathscr{T}_1(p) &= \cdots \overline{33} \cdot 1\bar{2} \cdot \bar{3} \cdot 3\bar{1} \cdot 2 \cdot \overline{33} \cdot \overline{33} \cdots \\
\mathscr{T}_2(p) &= \cdots \overline{33} \cdot 1\bar{2} \cdot \bar{3} \cdot \overline{21} \cdot \bar{3} \cdot 22 \cdot \overline{33} \cdots \\
\mathscr{T}_3(p) &= \cdots \overline{33} \cdot \overline{32} \cdot 1 \cdot \overline{21} \cdot \bar{3} \cdot 22 \cdot \overline{33} \cdots \\
\mathscr{T}_4(p) &= \cdots \overline{33} \cdot \overline{32} \cdot 1 \cdot \overline{32} \cdot \bar{1} \cdot 22 \cdot \overline{33} \cdots \\
\mathscr{T}_5(p) &= \cdots \overline{33} \cdot \overline{33} \cdot 1 \cdot \overline{32} \cdot \bar{1} \cdot 23 \cdot \overline{33} \cdots = T(p)
\end{aligned}$$

Regarding $\bar{3}$ as empty space in a box, one can interpret the above patterns as a motion of particles and anti-particles which can form a neutral (weight 0) bound state. We hope to report on the explicit algorithm for general $\hat{\mathfrak{g}}_n$ elsewhere.

A. PARAMETERIZATION OF B_l

We list the parameterization of the crystal B_l. In $A_n^{(1)}$ case, it may be identified with the set of semistandard Young tableaux of length l one row shape on letters $\{1,..., n+1\}$. For the other $\hat{\mathfrak{g}}_n$, B_l may be viewed as a similar set with some additional constraints. The relevant letters are $\{1,..., n, \bar{n},..., \bar{1}\}$ as well as 0 and/or $\emptyset$ depending on $\hat{\mathfrak{g}}_n$. The crystal structure (actions of $\tilde{e}_i$, $\tilde{f}_i$) can be found in the article[7] for $C_n^{(1)}$ and the ones[10, 12] for the other $\hat{\mathfrak{g}}_n$.

$$A_n^{(1)}\colon B_l = \left\{ (x_1,..., x_{n+1}) \in \mathbb{Z}^{n+1} \;\middle|\; x_i \geqslant 0, \sum_{i=1}^{n+1} x_i = l \right\}$$

$$A_{2n-1}^{(2)}\colon B_l = \left\{ (x_1,..., x_n, \bar{x}_n,..., \bar{x}_1) \in \mathbb{Z}^{2n} \;\middle|\; x_i, \bar{x}_i \geqslant 0, \sum_{i=1}^{n} (x_i + \bar{x}_i) = l \right\}$$

$$A_{2n}^{(2)}\colon B_l = \left\{ (x_1,..., x_n, \bar{x}_n,..., \bar{x}_1) \in \mathbb{Z}^{2n} \;\middle|\; x_i, \bar{x}_i \geqslant 0, \sum_{i=1}^{n} (x_i + \bar{x}_i) \leqslant l \right\}$$

We set $x_0 = l - \sum_{i=1}^n (x_i + \bar{x}_i)$.

$$B_n^{(1)}\colon B_l = \left\{ (x_1,..., x_n, x_0, \bar{x}_n,..., \bar{x}_1) \in \mathbb{Z}^{2n} \times \{0, 1\} \;\middle|\; \begin{array}{l} x_0 = 0 \text{ or } 1,\ x_i, \bar{x}_i \geqslant 0, \\ x_0 + \sum_{i=1}^n (x_i + \bar{x}_i) = l \end{array} \right\}$$

$$C_n^{(1)}\colon B_l = \left\{ (x_1,..., x_n, \bar{x}_n,..., \bar{x}_1) \in \mathbb{Z}^{2n} \;\middle|\; \begin{array}{l} x_i, \bar{x}_i \geqslant 0, \\ l \geqslant \sum_{i=1}^n (x_i + \bar{x}_i) \in l - 2\mathbb{Z} \end{array} \right\}$$

We set $x_0 = (l - \sum_{i=1}^n (x_i + \bar{x}_i))/2$.

$$D_n^{(1)}\colon B_l = \left\{ (x_1,..., x_n, \bar{x}_n,..., \bar{x}_1) \in \mathbb{Z}^{2n} \;\middle|\; \begin{array}{l} x_n = 0 \text{ or } \bar{x}_n = 0,\ x_i, \bar{x}_i \geqslant 0, \\ \sum_{i=1}^n (x_i + \bar{x}_i) = l \end{array} \right\}$$

$$D_{n+1}^{(2)}\colon B_l = \left\{ (x_1,..., x_n, x_0, \bar{x}_n,..., \bar{x}_1) \in \mathbb{Z}^{2n} \times \{0, 1\} \;\middle|\; \begin{array}{l} x_0 = 0 \text{ or } 1,\ x_i, \bar{x}_i \geqslant 0 \\ x_0 + \sum_{i=1}^n (x_i + \bar{x}_i) \leqslant l \end{array} \right\}$$

We set $x_\emptyset = l - x_0 - \sum_{i=1}^n (x_i + \bar{x}_i)$.

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